Analysis of a Simply Supported Laminated Anisotropic Rectangular Plate

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Using a Fourier series method, a solution is obtained to the governing equations of simply-supported laminated plates in which coupling occurs between bending and in-plane extension. Results are presented for bending under transverse load, natural frequencies of flexural vibrations, and buckling under uniform biaxial compression. Coupling is shown to reduce the effective stiffness of a laminate compared to analogous homogeneous orthotropic plates.

Nomenclature

E_{11},E_{22} = moduli of a unidirectional composite parallel and transverse to the direction of the fibers, respectively, lb/in.^{2}
F = E_{11}/E_{22} = stiffness ratio
G_{12} = longitudinal shear modulus of a unidirectional composite in 1-2 plane, lb/in.^{2}
h = plate thickness, in.
M_x . . . = M = distributed bending (and twisting) moments, lb
N_x . . . = N = stress resultants, lb/in.
P = number of layers
Q_{ij} = reduced stiffness matrix of a constituent layer, lb/in.^{2}
q = distributed load on surface of plate, lb/in.^{2}
R = a/b = length to width ratio of plate
t = time, sec
w = displacement of the plate in z direction, in.
ϵ_x^0 . . . = ϵ^0 = strain components of the middle surface, in./in.
κ_x . . . = κ = curvature of middle surface, 1/in.
ν_{12} = Poisson's ratio of unidirectional composite as determined from a tensile test parallel to the fibers
ρ = density, lb/in.^{2}
ϕ = stress function
ω = transverse vibration frequency, cycles/sec
θ = angle of 1-2 axes in constituent layer with respect to the x-y axes of the plate, rad

Introduction

In developing the equations of a laminated anisotropic plate, Reissner and Stavsky[1] showed the existence of a coupling phenomenon between bending and in-plane extension which is expressed in standard plate nomenclature by the constitutive relations

$$\begin{bmatrix} N \\ M \end{bmatrix} = \begin{bmatrix} A & B \\ B & D \end{bmatrix} \begin{bmatrix} \epsilon^0 \\ \kappa \end{bmatrix} \tag{1}$$

where

$$(A_{ij},B_{ij},D_{ij}) = \int_{-h/2}^{h/2} Q_{ij}(l,z,z^2)dz \quad (i,j = 1,2,6)$$

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The Q_{ij}'s are reduced stiffness coefficients as discussed by Ambartsumyan,[2] and by Tsai and Pagano.[3] Denoting differentiation by comma, the curvature-displacement relations are those found in homogeneous plate theory incorporating the Kirchhoff assumptions (i.e., the plate is assumed to be thin)

$$\kappa_x = -w_{,xx},\ \kappa_y = -w_{,yy},\ \kappa_{xy} = -2w_{,xy} \tag{2}$$

Similar results were obtained by Dong, Pister, and Taylor.[4] A more general treatment including in-plane force effects, thermal effects, and inertia terms has been developed by Whitney and Leissa.[5] Thus the theory of anisotropic laminated plates has been well established. However, solutions to boundary value problems for the coupled case ($B_{ij} \neq 0$) are very limited. Although some closed form solutions have been obtained[5,6] for coupled laminates with hinge-supports, solutions for the more classical boundary conditions (e.g., simple-supports, clamped, etc.) are nonexistent.

This paper is concerned with cross-ply and angle-ply laminates. Cross-ply laminates consist of an even number of layers all of the same thickness with the orthotropic axes of symmetry in each ply alternately oriented at angles of 0° and 90° to the plate axes. Angle-ply laminates also consist of an even number of layers having the same thickness; however, the orthotropic axes of symmetry in each ply are alternately oriented at angles of $+\theta$ and $-\theta$ to the plate axes. In particular, the purpose of this study is to obtain deflections due to uniform transverse load, natural frequencies of transverse vibration, and critical buckling loads for simply-supported cross-ply and angle-ply plates.

For a plate which is simply supported in the classical sense the plane portion of the problem involves stress boundary conditions. Thus it is convenient to write the governing equations (equilibrium and compatibility) in terms of a stress function ϕ and transverse displacement w. This can be accomplished by using a partially inverted form of the constitutive relations (1)

$$\begin{bmatrix} \epsilon^0 \\ M \end{bmatrix} = \begin{bmatrix} A^* & B^* \\ -(B^*)^T & D^* \end{bmatrix} \begin{bmatrix} N \\ \kappa \end{bmatrix} \tag{3}$$

where

$$A^* = A^{-1},\ B^* = A^{-1}B,\ D^* = D - BA^{-1}B$$

$$N_x = \phi_{,yy},\ N_y = \phi_{,xx},\ N_{xy} = -\phi_{,xy}$$

Bending under Uniform Transverse Load

For both cross-ply and angle-ply plates

$$A_{16} = A_{26} = D_{16} = D_{26} = 0 \tag{4}$$

Furthermore, in the case of cross-ply plates it can be shown that

$$A_{22} = A_{11},\ B_{22} = -B_{11},\ D_{22} = D_{11} \tag{5}$$

and that all other elements of the B matrix vanish. The governing equations are[4]

$$A_{11}{}^*\phi_{,xxxx} + L_1\phi + L_2 w = 0 \tag{6}$$

$$L_3 w + D_{11}{}^* w_{,yyyy} + L_2\phi = q \tag{7}$$

where the operators L_i are defined as follows

$$L_1 = (2A_{12}{}^* + A_{66}{}^*)(\ \ )_{,xxyy} + A_{11}{}^*(\ \ )_{,yyyy}$$

$$L_2 = B_{12}{}^*[(\ \ )_{,xxxx} - (\ \ )_{,yyyy}]$$

$$L_3 = D_{11}{}^*(\ \ )_{,xxxx} + 2(D_{12}{}^* + 2D_{66}{}^*)(\ \ )_{,xxyy}$$

For angle-ply laminates, B_{16} and B_{26} are the only nonvanishing elements of the B matrix, and the governing equations are[1,4]

$$A_{22}{}^*\phi_{,xxxx} + L_1\phi + L_4 w = 0 \tag{8}$$

$$L_3 w + D_{22}{}^* w_{,yyyy} - L_4\phi = q \tag{9}$$

where

$$L_4 = (B_{61}{}^* - 2B_{26}{}^*)(\ \ )_{,xxxy} + (B_{26}{}^* - 2B_{16}{}^*)(\ \ )_{,xyyy}$$

Consider a rectangular simply-supported cross-ply plate of dimensions a, b (as shown in Fig. 1) under a uniform transverse load $q = q_0$. The boundary conditions are

$$w(0,y) = w(a,y) = w(x,0) = w(x,b) = 0 \tag{10}$$

$$M_x(0,y) = M_x(a,y) = M_y(x,0) = M_y(x,b) = 0 \tag{11}$$

$$N_x(0,y) = N_x(a,y) = N_y(x,0) = N_y(x,b) = 0 \tag{12}$$

$$N_{xy}(0,y) = N_{xy}(a,y) = N_{xy}(x,0) = N_{xy}(x,b) = 0 \tag{13}$$

Solutions to Eqs. (6) and (7) are assumed to be of the form

$$\phi = \sum_{m=1}^{\infty}\sum_{m=1}^{\infty} A_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \tag{14}$$

$$w = \sum_{m=1}^{\infty}\sum_{m=1}^{\infty} B_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \tag{15}$$

Equation (15) will satisfy the boundary conditions (10–13). However, Eq. (14) will not satisfy the conditions involving N_{xy}. In order to satisfy (13) a procedure used by Green[7] on homogeneous, isotropic plates is applied. In particular, Eq. (14) cannot be differentiated term-by-term beyond the second order in x and y. Thus,

$$\phi_{,xx} = -\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{m^2\pi^2}{a^2}A_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \tag{16}$$

$$(0 < x < a,\ 0 \leq y \leq b)$$

$$\phi_{,xxx} = -\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}A'_{mn}\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \tag{17}$$

$$(0 \leq x \leq a,\ 0 \leq y \leq b)$$

A'_{mn} can be related to A_{mn} by a partial integration of Eq. (17) with the result (see the Appendix)

$$\phi_{,xxx} = \frac{-1}{2}\sum_{n=1}^{\infty}a_n\sin\frac{n\pi y}{b} - \sum_{m=2,4,\ldots}^{\infty}\sum_{n=1}^{\infty}\times \left(\frac{m^3\pi^3}{a^3}A_{mn} + a_n\right)\cos\frac{m\pi x}{a}\sin\frac{m\pi y}{b} - \sum_{m=1,3,\ldots}^{\infty}\sum_{n=1}^{\infty}\times \left(\frac{m^3\pi^3}{a^3}A_{mn} + b_n\right)\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \tag{18}$$

where

$$a_n = \frac{4}{ab}\int_0^b[\phi_{,xx}(a,y) - \phi_{,xx}(0,y)]\sin\frac{n\pi y}{b}dy$$

$$b_n = -\frac{4}{ab}\int_0^b[\phi_{,xx}(a,y) + \phi_{,xx}(0,y)]\sin\frac{n\pi y}{b}dy$$

Thus, a_n and b_n are coefficients in a Fourier expansion of $\phi_{,xx}$ on the boundary.

Proceeding in a similar manner we find

$$\phi_{,yyy} = \frac{-1}{2}\sum_{m=1}c_m\sin\frac{m\pi x}{a} - \sum_{m=1}^{\infty}\sum_{n=2,4,\ldots}^{\infty}\times \left(\frac{n^3\pi^3}{b^3}A_{mn} + c_m\right)\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b} - \sum_{m=1}^{\infty}\sum_{n=1,3,\ldots}^{\infty}\times \left(\frac{n^3\pi^3}{b^3}A_{mn} + d_m\right)\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b} \tag{19}$$

where c_m and d_m are analogous to a_n and b_n. All other desired derivatives of ϕ can be obtained from a term-by-term differentiation.

Substituting the appropriate derivatives of ϕ and w into Eqs. (6) and (7), expanding q_0 in a double Fourier sine series, and solving the resulting simultaneous equations yields

$$A_{mn} = [-b^3R^3/\pi^3C_{mn}][(A_{11}{}^*G_{mn} + B_{12}{}^*F_{mn})mb_n + (A_{11}{}^*G_{mn} - B_{12}{}^*F_{mn})nRd_m + 16q_0RbF_{mn}/\pi^3mn] \tag{20}$$

$$B_{mn} = [b^3R^3/\pi^3h^2C_{mn}][(B_{12}{}^*E_{mn} - A_{11}{}^*F_{mn})\times (mb_n - nRd_m) + 16q_0R_bE_{mn}/\pi^3mn] \tag{21}$$

where

$$E_{mn} = A_{11}{}^*(m^4 + n^4R^4) + (2A_{12}{}^* + A_{66}{}^*)m^2n^2R^2$$

$$F_{mn} = B_{12}{}^*(m^4 - n^4R^4)$$

$$G_{mn} = D_{11}{}^*(m^4 + n^4R^4) + 2(D_{12}{}^* + 2D_{66}{}^*)m^2n^2R^2$$

$$C_{mn} = E_{mn}G_{mn} + F_{mn}{}^2$$

In order for the shear resultant N_{xy} to vanish on the boundary,

$$\sum_{n=1,3,\ldots}^{\infty} nA_{mn} = 0 \tag{22}$$

for $m = 1,3,\ldots$ and

$$\sum_{m=1,3,\ldots}^{\infty} mA_{mn} = 0 \tag{23}$$

for $n = 1,3,\ldots$.

Substituting (20) into (22) and (23) leads to the following infinite set of simultaneous equations:

$$m\sum_{m=1,3,\ldots}^{\infty}\frac{(A_{11}{}^*G_{mn} + B_{12}{}^*F_{mn})b_n}{C_{mn}} + Rd_m\sum_{n=1,3,\ldots}^{\infty}\frac{(A_{11}{}^*G_{mn} - B_{12}{}^*F_{mn})n}{C_{mn}} = \frac{-16q_0Rb}{\pi m}\sum_{n=1,3,\ldots}^{\infty}\frac{F_{mn}}{mC_{mn}} \tag{24}$$

$$b_n\sum_{m=1,3,\ldots}^{\infty}\frac{(A_{11}{}^*G_{mn} + B_{12}{}^*F_{mn})m}{C_{mn}} + nR\sum_{m=1,3,\ldots}^{\infty}\frac{(A_{11}{}^*G_{mn} - B_{12}{}^*F_{mn})d_m}{C_{mn}} = \frac{-16q_0Rb}{\pi n}\sum_{m=1,3,\ldots}^{\infty}\frac{F_{mn}}{mC_{mn}} \tag{25}$$

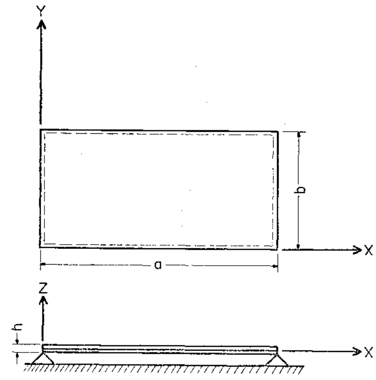

Fig. 1 Simply supported plate.

Equations (24) and (25) are truncated to obtain a finite set of equations. The nonhomogeneous terms and the coefficients of b_n and d_m which are an infinite series should be summed separately. Although the results cannot be expressed in closed form without considerable difficulty (if at all), sufficient terms can be summed to obtain a desired degree of convergence. The truncated equations were solved on an IBM 7094 digital computer. Convergence was established by increasing the order of the system. Stresses and moments can be determined from the constitutive relations (3).

Now consider a simply-supported angle-ply plate under the uniform load q_0. The boundary conditions are the same as for the cross-ply plate. Solutions to Eqs. (8) and (9) are assumed to be of the form

$$\phi = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} H_{mn} \cos\frac{m\pi x}{a} \cos\frac{n\pi y}{b} \tag{26}$$

$$w = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} J_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \tag{27}$$

Equation (26) will not satisfy the condition that N_x and N_y vanish on the boundary. Again using Green's method ϕ can be differentiated term-by-term through the third order in x and y. Using the same procedure as for cross-ply composites we find

$$H_{mn} = \frac{-R^4 b^4}{\pi^4 k_{mn}} \left[N_{mn}(A_{22}^* e_n + A_{11}^* f_m) + \frac{16 q_0 M_{mn}}{\pi^2 mn} \right] \tag{28}$$

$$J_{mn} = \frac{R^4 b^4}{\pi^2 k_{mn}} \left[M_{mn}(A_{22}^* e_n + A_{11}^* f_m) + \frac{16 q_0 L_{mn}}{\pi^2 mn} \right] \tag{29}$$

where

$$L_{mn} = A_{22}^* m^4 + A_{11}^* n^4 R^4 + (2A_{12}^* + A_{66}^*) m^2 n^2 R^2$$

$$M_{mn} = mnR[(B_{61}^* - 2B_{26}^*) m^2 + (B_{62}^* - 2B_{16}^*) n^2 R^2]$$

$$N_{mn} = D_{11}^* m^4 + 2(D_{12}^* + 2D_{66}^*) m^2 n^2 R^2 + D_{22}^* n^4 R^4$$

$$K_{mn} = L_{mn} N_{mn} - M_{mn}^2$$

$$e_n = \frac{-4}{ab} \int_0^b [\phi_{,xxx}(a,y) + \phi_{,xxx}(0,y)] \cos\frac{n\pi y}{b}\, dy$$

$$f_m = \frac{-4}{ab} \int_0^a [\phi_{,yyy}(x,b) + \phi_{,yyy}(x,0)] \cos\frac{m\pi x}{a}\, dx$$

The requirement that N_x and N_y vanish on the boundary leads to the following infinite set of simultaneous equations

$$A_{22}^* \sum_{n=1,3,\ldots}^{\infty} \frac{N_{mn}}{K_{mn}} e_n + A_{11}^* f_m \sum_{n=1,3,\ldots}^{\infty} \frac{N_{mn}}{K_{mn}} = \frac{-16 q_0}{\pi^2 m} \sum_{n=1,3,\ldots}^{\infty} \frac{M_{mn}}{n K_{mn}} \tag{30}$$

for $m = 1,3,\ldots$

$$A_{22}^* \sum_{m=1,3,\ldots}^{\infty} \frac{N_{mn}}{K_{mn}} + A_{11}^* \sum_{m=1,3,\ldots}^{\infty} \frac{N_{mn}}{K_{mn}} f_m = \frac{-16 q_0}{\pi^2 n} \sum_{m=1,3,\ldots}^{\infty} \frac{M_{mn}}{m K_{mn}} \tag{31}$$

for $n = 1,3,\ldots$.

The system of equations generated by (30) and (31) is solved in the same manner as described for cross-ply laminates.

Transverse Vibration

For dynamic problems in which in-plane and rotary inertia are neglected, Eqs. (7) and (9) become, in the absence of transverse load,[5]

$$L_3 w + D_{11}^* w_{,yyyy} + L_2 \phi + \rho w_{,tt} = 0 \tag{32}$$

$$L_3 w + D_{22}^* w_{,yyyy} - L_4 \phi + \rho w_{,tt} = 0 \tag{33}$$

respectively. Equations (6) and (8) are unchanged.

Natural frequencies can be obtained for both simply-supported cross-ply and angle-ply plates by multiplying Eqs. (14, 15, 26, and 27) by $e^{i\omega t}$. Using these functions in conjunction with the Fourier method previously described, substituting the results into Eqs. (6, 8, 32, and 33), and applying the boundary conditions (12) and (13) leads to four infinite sets of simultaneous algebraic equations for cross-ply and angle-ply plates, respectively. For cross-ply laminates the following equations are obtained

$$m \sum_n \frac{(A_{11}^* G'_{mn} + B_{12}^* F_{mn}) b_n}{C_{mn}} + R d_m \sum_n \frac{(A_{11}^* G'_{mn} - B_{12}^* F_{mn}) n}{C_{mn}} = 0 \tag{34}$$

$$b_n \sum_m \frac{(A_{11}^* G'_{mn} + B_{12}^* F_{mn})}{C_{mn}} + nR \sum_m \frac{(A_{11}^* G'_{mn} - B_{12}^* F_{mn}) d_m}{C_{mn}} = 0 \tag{35}$$

where

$$G'_{mn} = D_{11}^*(m^4 + n^4 R^4) + 2(D_{12}^* + 2D_{66}^*) m^2 n^2 R^2 \frac{-\rho \omega^2 R^4 b^4}{\pi^4}$$

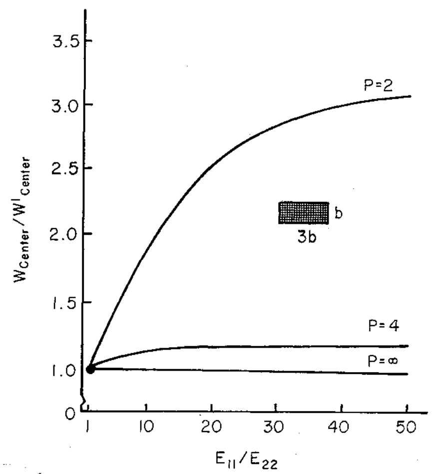

Fig. 2 Center deflection ratio of coupled cross-ply laminate to equivalent orthotropic plate as a function of E_{11}/E_{22} for uniform transverse load q_0.

Four sets of homogeneous equations (corresponding to combinations of symmetric and antisymmetric modes) are obtained from (34) and (35) by ranging m and n over even or odd positive integers.

In each group of these equations a nontrivial solution is obtained if ω is chosen such that the determinant of the coefficient matrix vanishes. As in the case of transverse loading, coefficients of b_n and d_m which are an infinite series should be summed separately.

For angle-ply laminates the following equations are obtained

$$A_{22}^* \sum_n \left(\frac{N'_{mn}}{K_{mn}}\right) e_n + A_{11}^* f_m \sum_n \left(\frac{N'_{mn}}{K_{mn}}\right) = 0 \quad (36)$$

$$A_{22}^* e_n \sum_m \left(\frac{N'_{mn}}{K_{mn}}\right) + A_{11}^* \sum_m \left(\frac{N'_{mn}}{K_{mn}}\right) f_m = 0 \quad (37)$$

where

$$N'_{nm} = D_{11}^* m^4 + 2(D_{12}^* + 2D_{66}^*) m^2 n^2 R^2 + D_{22} n^4 R^4 \frac{-\rho\omega^2 R^4 b^4}{\pi^4}$$

Buckling of Angle-Ply Laminates

For a buckling analysis it is necessary to include the effects of in-plane forces. Thus Eq. (9) becomes, in the absence of transverse load[5]

$$L_3 w + D_{22}^* w_{,yyyy} - L_4\phi = N_x^i w_{,xx} + 2N_{xy}^i w_{,xy} + N_y^i w_{,yy} \quad (38)$$

where superscript i denotes prebuckling (initial) values.

Consider an angle-ply composite subjected to uniform biaxial compression. The initial stress field becomes

$$N_x = -N_0 = \text{const}, \ N_y = -kN_0, \ N_{xy} = 0 \quad (39)$$

where k is a constant. In addition to conditions (10, 11, and 13) we require that

$$N_x(0,y) = N_x(a,y) = -N_0 \quad (40)$$

$$N_y(x,0) = N_y(x,b) = -kN_0 \quad (41)$$

To obtain a critical value of N_0 the stress function solution to Eqs. (8) and (32) is assumed to be of the form

$$\phi = \frac{-N_0}{2}(kx^2 + y^2) + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} H_{mn} \cos\frac{m\pi x}{a} \cos\frac{n\pi y}{b} \quad (42)$$

Equation (27) is assumed valid for w. Using the Fourier method in conjunction with Eqs. (27) and (42), substituting the results into Eqs. (8) and (32), and applying the boundary conditions (40) and (41) leads to Eqs. (36) and (37) with N'_{mn} replaced by N''_{mn} where

$$N''_{mn} = D_{11}^* m^4 + 2(D_{12}^* + 2D_{66}^*) m^2 n^2 R^2 + D_{22}^* n^4 R^4 - (N_0 R^2 b^2/\pi^2)(m^2 + kn^2R^2)$$

Again four sets of equations result and the critical N_0 is determined in the same manner as ω in the case of flexural vibrations. The lowest critical value of N_0 is the buckling load.

Discussion and Conclusions

If the coupling coefficients B_{ij} are neglected, Eqs. (6) and (8), and Eqs. (7, 9, 32, and 33) reduce to those of an orthotropic, homogeneous in-plane elasticity and plate bending problem, respectively. The relationships between the coupling coefficients and the properties of the individual plies are given by

$$B_{11} = [(F - 1)/4P]h^2 Q_{22}, \ (B_{16}, B_{26}) = (h^2/2P)(Q_{16}, Q_{26}) \quad (43)$$

Thus as the number of layers increases, the solution of an orthotropic, homogeneous plate becomes the limiting case. In addition, the effect of coupling depends on the degree of anisotropy of the individual layers (i.e., E_{11}/E_{22}). These conclusions are illustrated in Figs. 2–6 where the ratio of the coupled solution to the orthotropic solution (denoted by primes) is plotted as a function of E_{11}/E_{22} for bending de-

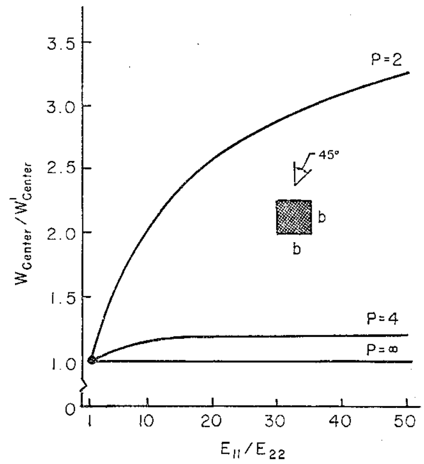

Fig. 4 Center deflection ratio of coupled ±45° angle-ply laminate to equivalent orthotropic plate as a function of E_{11}/E_{22} for uniform transverse load q_0.

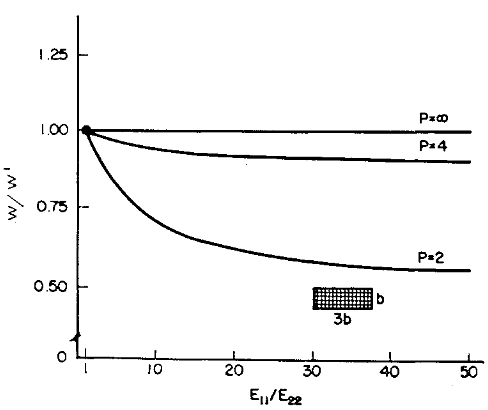

Fig. 3 Fundamental vibration frequency ratio of coupled cross-ply laminate to equivalent orthotropic plate as a function of E_{11}/E_{22}.

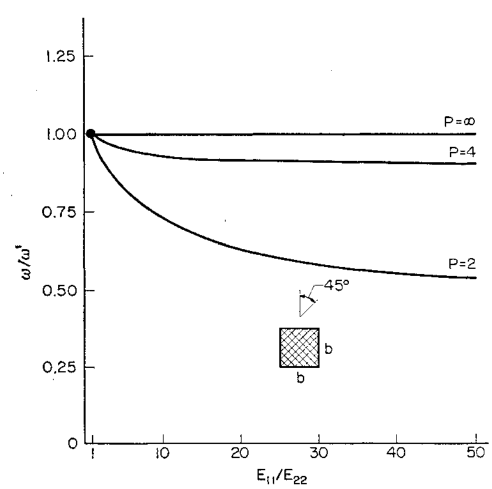

Fig. 5 Fundamental vibration frequency ratio of coupled ±45° angle-ply laminate to equivalent orthotropic plate as a function of E_{11}/E_{22}.

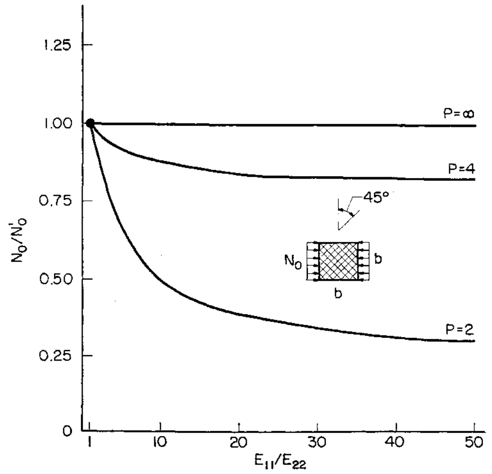

Fig. 6 Critical buckling load ratio of coupled ±45° angle-ply laminate to equivalent orthotropic plate as a function of E_{11}/E_{22} for uniform axial compression N_0.

flections, fundamental vibration frequencies, and buckling loads. Figures 2 and 3 are for cross-ply laminates, having an aspect ratio of 3, whereas Figs. 4–6 are for square, ±45°, angle-ply plates. In all cases ν_{12} is assumed to be 0.25. For the cross-ply plates $G_{12}/E_{22} = 1$ and for angle-ply plates $G_{12}/E_{22} = \frac{1}{2}$. These numbers are typical of materials such as fiberglass, boron-epoxy fibrous composites, graphite-epoxy fibrous composites, and plywood. Numerical results are rather insensitive to small changes in the above parameters. Thus the ratio E_{11}/E_{22}, which has a wide range of variation for practical engineering laminates, is the parameter of primary interest.

For all of the cases shown the effect of coupling is severe for two-layer composites. The effect dissipates rather rapidly as the number of layers are increased. The effect of coupling is also directly proportional to E_{11}/E_{22}. The orthotropic solutions ($B_{ij} = 0$) corresponding to Figs. 2–6 are shown in Table 1. It should be noted that the plate thickness is kept constant as the number of layers are increased. For all of the results obtained the general Fourier method converges quite rapidly.

A cursory examination of the numerical results reveals that coupling reduces the effective plate stiffness. That is, transverse deflections are increased while fundamental vibration frequencies and critical buckling loads are decreased compared to analogous orthotropic, homogeneous plates. This observation led Ashton[8] to investigate an approximate method for solving coupled plate problems which is referred to as the "reduced bending stiffness method." Specifically, one simply ignores the B^* coupling terms in the partially inverted form of the constitutive relationship (3). This procedure leads to the equations of an orthotropic, homogeneous plate with the bending stiffness D_{ij} replaced by D_{ij}^*. If this procedure is valid then the solution to coupled plate problems should be independent of the in-plane boundary conditions. Thus the results obtained in this work for simple-support conditions should be the same as those obtained in Refs. 5 and 6 for hinge-support conditions. That is, conditions (10) and (11) are applied in both cases. Table 2 shows a comparison between the results obtained for hinge-free normal-supports (HFN), hinge-free tangential-supports (HFT), simple-supports (SS), and the reduced bending stiffness method (RBS) for cross-ply plates and angle-ply plates under uniform transverse load. In all cases the reduced bending stiffness method agrees very well with the simple-support results. These results also compare favorably with those obtained for hinge-supports except for the ±15° angle-ply plate where the difference is greater than 20%. Similar comparison for flexural vibrations and buckling shows good agreement between the three different solutions However, because of the results obtained for ±15° angle-ply laminates under transverse load and the limited number of boundary conditions investigated, indiscriminate use of the RBS method should be discouraged.

The general Fourier method discussed in this paper can be used to solve problems having other edge conditions on coupled laminated plates. For displacement or mixed boundary conditions the governing equations should be in terms of displacements as presented in Ref. 5.

Appendix: Fourier Analysis

When using a double Fourier series to represent the solution of a boundary value problem, it is inherently assumed that the series is term by term differentiable. It has been shown by Green[7] that the validity of such an assumption depends on the boundary conditions satisfied by the function.

Let $f(x,y)$ be a function which can be expanded in a double sine series over the region $0 < x < a$, $0 < y < b$. If the partial derivative $f_{,x}$ can be expanded in a cosine-sine series then the coefficients can be related to the original series by a partial integration. Similar statements apply when f is expanded in a sine-cosine series, a cosine-cosine series or a cosine-sine series and correspondingly $f_{,x}$ is expanded in a cosine-cosine series, a sine-cosine series or a sine-sine series. Thus,

$$f(x,y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \qquad \text{(A1)}$$

$$(0 < x < a,\ 0 < y < b)$$

$$f_{,x} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} B_{mn} \cos\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \qquad \text{(A2)}$$

$$(0 \leq x \leq a,\ 0 < y < b)$$

then

$$B_{on} = \frac{2}{ab}\int_0^b [f(a,y) - f(0,y)] \sin\frac{n\pi y}{b}\, dy \qquad \text{(A3)}$$

$$B_{mn} = \frac{m\pi}{a} A_{mn} + \frac{4}{ab}\int_0^b [f(a,y) - f(0,y)] \sin\frac{n\pi y}{b}\, dy \qquad \text{(A4)}$$

$$(m \text{ even},\ m \neq 0)$$

$$B_{mn} = \frac{m\pi}{a} A_{mn} - \frac{4}{ab}\int_0^b [f(a,y) + f(0,y)] \sin\frac{n\pi y}{b}\, dy \qquad \text{(A5)}$$

$$(m \text{ odd})$$

Let us assume the following functions can be expanded in a

Table 1 Orthotropic solution ($P = \infty$)

E_{11}/E_{22}	w'_{center} $(E_{22}h^3/q_0b^4) \times 10^2$	$\omega b^2(\rho/E_{22}h^3)^{1/2}$	$N_{cr}b^2/E_{22}/h^3$
Cross ply ($R = 3$)			
1	13.8	3.59	...
5	5.30	5.39	...
10	2.97	7.04	...
20	1.57	9.52	...
30	1.07	11.5	...
40	0.811	13.2	...
50	0.653	14.6	...
Angle ply ($R = 1, \theta = 45°$)			
1	4.57	5.88	3.51
5	1.59	9.93	9.99
10	0.873	13.4	18.2
20	0.458	18.5	34.7
30	0.312	22.5	51.1
40	0.235	25.8	67.5
50	0.189	28.8	84.0

Table 2 Comparison of center deflections for 2-layer laminates under uniform transverse load

Cross-ply laminates ($E_{11}/E_{22} = 40$, $P = 2$) ($w_{\text{center}}E_{22}h^3/q_0b^4$) $\times 10^2$			
R	HFN	SS	RBS
1	1.1249	1.0874	1.0874
3	2.4364	2.4354	2.4368
5	2.3555	2.3728	2.3799
Angle-ply laminates ($E_{11}/E_{22} = 40$, $P = 2$, $R = 1$) ($w_{\text{center}}E_{22}h^3/q_0b^4$) $\times 10^3$			
θ	HFT	S	RBS
15°	7.1416	8.7934	8.8268
30°	7.7576	8.0678	8.0723
45°	7.3217	7.3241	7.3245

Fourier series:

$$\frac{2[f(a,y) - f(0,y)]}{a} = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi y}{b} \qquad (0 < y < b) \tag{A6}$$

$$\frac{-2[f(a,y) + f(0,y)]}{a} = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi y}{b} \qquad (0 < y < b) \tag{A7}$$

Then from the theory of Fourier series we have

$$a_n = \frac{4}{ab}\int_0^b [f(a,y) - f(0,y)] \sin\frac{n\pi y}{b}\, dy \tag{A8}$$

$$b_n = -\frac{4}{ab}\int_0^b [f(a,y) + f(0,y)] \sin\frac{n\pi y}{b}\, dy \tag{A9}$$

Thus

$$B_{on} = \frac{a_n}{2} \tag{A10}$$

$$B_{mn} = \frac{m\pi}{a} A_{mn} + a_n \quad (m \text{ even}, m \neq 0) = \frac{m\pi}{a} A_{mn} + b_n \quad (m \text{ odd}) \tag{A11}$$

Thus term by term differentiability depends on the boundary conditions satisfied by $f(x,y)$. In particular, if $f(x,y)$ vanishes on the boundary then the series (A1) is term by term differentiable. Similarly, if

$$f(x,y) = \sum_{m=1}^{\infty}\sum_{n=0}^{\infty} A_{mn} \sin\frac{m\pi x}{a} \cos\frac{n\pi y}{b} \qquad (0 < x < a, 0 \leq y \leq b) \tag{A12}$$

and

$$f_{,x} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} B_{mn} \cos\frac{m\pi x}{a} \cos\frac{n\pi y}{b} \qquad (0 \leq x \leq a, 0 \leq y \leq b) \tag{A13}$$

then

$$B_{00} = \tfrac{1}{4}c_0 \tag{A14}$$

$$B_{mo} = (m\pi/a)A_{mo} + \tfrac{1}{2}c_0 \quad (m \text{ even}, m \neq 0) = (m\pi/a)A_{mo} + \tfrac{1}{2}d_o \quad (n \text{ even}) \tag{A15}$$

$$B_{on} = \tfrac{1}{2}c_n \quad (n \neq 0) \tag{A16}$$

$$B_{mn} = (m\pi/a)A_{mn} + c_n \quad (m \text{ even}, m \neq 0) = (m\pi/a)A_{mn} + d_n \quad (m \text{ odd}) \tag{A17}$$

If

$$f(x,y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} A_{mn} \cos\frac{m\pi x}{a} \cos\frac{n\pi y}{b} \qquad (0 \leq x \leq a, 0 \leq y \leq b) \tag{A18}$$

then

$$f_{,x} = -\sum_{m=1}^{\infty}\sum_{n=0}^{\infty} \frac{m\pi}{a} A_{mn} \sin\frac{m\pi x}{a} \cos\frac{n\pi y}{b} \qquad (0 < x < a, 0 \leq y \leq b) \tag{A19}$$

or if

$$f(x,y) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} A_{mn} \cos\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \qquad (0 \leq x \leq a, 0 < y < b) \tag{A20}$$

then

$$f_{,x} = -\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{m\pi}{a} A_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \qquad (0 < x < a, 0 < y < b) \tag{A21}$$

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